MATH 2060 Mathematical Analysis II
Tutorial Class 2
Lee Man Chun

1. (a) State Mean Value Theorem.
   (b) Let $f : \mathbb{R} \to \mathbb{R}$ be a function differentiable on $\mathbb{R}$. prove that if f' is bounded on $\mathbb{R}$, then f is uniformly continuous.
   (c) Let $f : (a,b) \to \mathbb{R}$ be a differentiable function such that f' is bounded on (a,b). Show that f is bounded function.
   (d) If f is uniform continuous on $[a,b]$ and differentiable on (a,b) , is f' bounded on (a,b) ? Prove or disprove it.

2. (a) Let $f : [a,b] \to \mathbb{R}$ be a function continuous on $[a,b]$ and differentiable on (a,b). If $f' > 0$ on (a,b), show that f is strictly increasing on $[a,b]$.
   (b) Prove that $\tan x > x > \sin x > \frac{2}{\pi}x$ for all $x \in (0, \frac{\pi}{2})$.
   (c) Let $f : [a,b] \to \mathbb{R}$ be a function continuous on $[a,b]$ and differentiable on (a,b). Show that if $\lim_{x \to a} f'(x) = A$, then $f'(a)$ exists and equals to A.

3. Let $f : \mathbb{R} \to \mathbb{R}$be a twice differentiable function. Suppose

$$f(x) \leq 0 \quad \text{and} \quad f''(x) \geq 0 \quad , \forall x \in \mathbb{R}.$$

   Prove that f is constant function.

4. Let $f : [0,\infty) \to \mathbb{R}$ be a differentaible function on $(0,+\infty)$ and assume $\lim_{x \to \infty} f'(x) = b$.

   (a) Show that for any $h > 0$, we have $\lim_{x \to \infty} \frac{f(x+h) - f(x)}{h} = b$.
   (b) Show that if $f(x) \to a$ as $x \to \infty$, then $b = 0$.
   (c) Show that $\lim_{x \to \infty} \frac{f(x)}{x} = b$.

5. (a) State the Taylor's theorem.
   (b) Prove that $\sin x < x - \frac{x^3}{6} + \frac{x^5}{120}$ for all $x \in (0, \pi]$.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely many times differentiable function satisfying
   (i) $f(x) > f(0)$ for all $x \neq 0$, and
   (ii) there exists $M > 0$ such that $|f^{(n)}(x)| \leq M$ for all $x \in \mathbb{R}, n \in \mathbb{N}$.

   (a) Show that there exists $n \in \mathbb{N}$ such that $f^{(n)}(0) \neq 0$.
   (b) Prove that there exists an even number $2k$ such that $f^{(2k)}(0) > 0$.
   (c) Prove that there exists $\delta > 0$ such that $f'(y) < 0 < f'(x)$ for all x, y with $-\delta < y < 0 < x < \delta$.

MATH 2060 Mathematical Analysis II
Tutorial Class 2
Lee Man Chun

1. Evaluate the Limits:
   (a) $\lim_{x\to 1^+} x^{\frac{1}{x-1}}$
   (b) $\lim_{x\to 0^+} \frac{e^{\frac{-1}{x}}}{x}$

2. Let $I \subset \mathbb{R}$ be an open interval, let $f : I \to \mathbb{R}$ be differentiable on I, and suppose $f''(a)$ exists at $a \in I$. Show that

$$f''(a) = \lim_{h\to 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2}$$

   Give an example where this limit exists, but the function is not twice differentiable at a.

3. Suppose the function $f : (-1,1) \to R$ has n derivatives, and $f^{(n)} : (-1,1) \to \mathbb{R}$ is bounded. Prove that there exists $M > 0$ such that $|f(x)| \leq M|x|^n, \forall x \in (-1,1)$ if and only if $f(0) = f'(0) = ... = f^{(n-1)}(0) = 0$.

4. (a) State the Taylor's theorem.
   (b) Prove that $\sin x < x - \dfrac{x^3}{6} + \dfrac{x^5}{120}$ for all $x \in (0,\pi]$.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely many times differentiable function satisfying
   (i) $f(x) > f(0)$ for all $x \neq 0$, and
   (ii) there exists $M > 0$ such that $|f^{(n)}(x)| \leq M$ for all $x \in \mathbb{R}, n \in \mathbb{N}$.
   (a) Show that there exists $n \in \mathbb{N}$ such that $f^{(n)}(0) \neq 0$.
   (b) Prove that there exists an even number $2k$ such that $f^{(2k)}(0) > 0$.
   (c) Prove that there exists $\delta > 0$ such that $f'(y) < 0 < f'(x)$ for all x, y with $-\delta < y < 0 < x < \delta$.

MATH 2060 Mathematical Analysis II
Tutorial Class 2
Lee Man Chun

1. (a) Dene Riemann integrability of a function.
   (b) Let $f : [a, b] \to \mathbb{R}$ be a Riemann integrable. Prove that f is bounded.
   (c) Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Show that f is Riemann integrable if and only if there exists exactly one value A such that

   $$L(f, P) \leq A \leq U(f, P) \text{ for every partition } P \text{ of the interval } [a, b].$$

   (d) Let $f : [a, b] \to \mathbb{R}$ be a bounded function.
   Show that f is Riemann integrable if and only if for all $\epsilon > 0$, there exists a partition P of $[a, b]$ such that

   $$U(f, P) - L(f, P) < \epsilon.$$

2. Show that any continuous function $f : [a, b] \to \mathbb{R}$ is integrable.

3. Are the following functions integrable ?
   (a) Let $f : [0, 1] \to \mathbb{R}$,

   $$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ -x & \text{otherwise.} \end{cases}$$

   (b) Let $f : [0, 1] \to \mathbb{R}$.

   $$f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n}, \text{ for some } n \in \mathbb{N} \\ g(x) & \text{otherwise.} \end{cases}$$

   where $g : [0, 1] \to \mathbb{R}$ is a continuous function.

4. Suppose that a integrable function $f : [a, b] \to \mathbb{R}$ has the property that $f(x) \geq 0$, $\forall x \in [a, b]$. Prove that $\int_b^a f \geq 0$.

5. If $f : [a, b] \to \mathbb{R}$ is a integrable function and $f(x) = C$, $\forall x \in \mathbb{Q} \cap [0, 1]$. Find $\int_b^a f$.

MATH 2060 Mathematical Analysis II
Tutorial Class 5
Lee Man Chun

1. (a) Show that if $f \in R[a,b]$, then for any sequence of tagged partition $\dot{P}_n$ of $[a,b]$, $||P_n|| \to 0$ implies $S(f,\dot{P}_n) \to \int_a^b f$ as $n \to \infty$.

   (b) Find the following limits.

   i. $\lim\limits_{n\to\infty} \sum\limits_{k=0}^{n} \frac{1}{k+n}$.

   ii. $\lim\limits_{n\to\infty} \left[\frac{n^2}{n^2+1} \cdot \frac{n^2}{n^2+2^2} \cdot \frac{n^2}{n^2+3^2} \cdots \frac{n^2}{n^2+n^2} \right]^{\frac{1}{n}}$.

2. Let $f : [a,b] \to \mathbb{R}$ be Riemann integrable and $g : \mathbb{R} \to \mathbb{R}$ be continuous. Show that $g \circ f$ is Riemann integrable on $[a,b]$.

3. Let $f : [a,b] \to \mathbb{R}$ be a bounded function at which $f \in R[c,b]$ for any $c > a$. Prove that $f \in R[a,b]$ and $\displaystyle\int_a^b f = \lim_{c\to a^+} \int_c^b f$.

4. (a) Let $g \in R[a,b]$ and $f : [a,b] \to \mathbb{R}$ be a continuous function. Suppose $g \geq 0$ on $[a,b]$. Show that there exists $c \in [a,b]$ such that $\displaystyle\int_a^b fg = f(c)\int_a^b g$.

   (b) Let $f : [0,+\infty) \to \mathbb{R}$ be a continuous function with $\lim_{x\to\infty} f(x) = L \in \mathbb{R}$. Suppose $\{a_n\}, \{b_n\}$ are two sequence in $\mathbb{R}^+$ such that $a_n \to 0, b_n \to \infty$ as $n \to \infty$. Show that for all $0 < r < s$,

$$\lim_{n\to\infty} \int_{a_n}^{b_n} \frac{f(rx) - f(sx)}{x} = (f(0) - L) \log \frac{s}{r}.$$

5. Let $f : [a,b] \to \mathbb{R}$ be a continuous function at which $f(x) \geq 0$. Show that

$$\lim_{n\to\infty} (\int_a^b f^n)^{\frac{1}{n}} = \sup\{f(x) : x \in [a,b]\}.$$

MATH 2060 Mathematical Analysis II
Tutorial Class 6

Lee Man Chun

Theorem 1 (The second fundamental theorem of Calculus)**.** *Suppose that the function* $f : [a,b] \to \mathbb{R}$ *is continuous. Then* $F(x) = \displaystyle\int_a^x f$ *satisfy*

$$F'(x) = f(x) \ , \forall x \in (a,b).$$

Problems :

1. (a) Prove the Second Fundamental Theorem of Calculus.

   (b) State and prove the Integration by Parts formula.

   (c) Let $f : [0, +\infty) \to \mathbb{R}$ be a continuous function. Define

   $$F(x) = \int_0^x f(x^2 + y)dy.$$

   Find $F'(x)$.

2. Let $f : [a,b] \to \mathbb{R}$ be a continuous function at which $f(x) \geq 0$. Show that

   $$\lim_{n\to\infty} (\int_a^b f^n)^{\frac{1}{n}} = \sup\{f(x) : x \in [a,b]\}.$$

3. Suppose that the function $f : [a,b] \to \mathbb{R}$ is continuous and it is twice differentiable on (a,b). Prove that there is a point $\eta \in (a,b)$ at which

   $$\int_a^b f(x)dx = \frac{b-a}{2}\left[f(a) + f(b)\right] - \frac{(b-a)^3}{12} f''(\eta).$$

4. (a) Suppose $f : [0, +\infty) \to \mathbb{R}$ is continuous and strictly increasing, and that $f : (0, +\infty] \to \mathbb{R}$ is differentiable and $f(0) = 0$. Prove that for all $a > 0$,

   $$\int_0^a f + \int_0^{f(a)} f^{-1} = af(a).$$

   (b) If f satisfies the assumption above, prove that for all $a > 0$ and $b > 0$,

   $$\int_0^a f + \int_0^b f^{-1} \geq ab$$

   (c) If a and b are two non-negative real number, p and q are positive real number such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$, show that

   $$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

MATH 2060 Mathematical Analysis II
Tutorial Class 7
Lee Man Chun

1. Let f, g be continuous function defined on $[a, b]$. Suppose that $f(x) \geq g(x)$ for all $x \in [a, b]$ and $g(x) \neq f(x)$. Show that
$$\int_a^b f > \int_a^b g.$$

2. (a) Define the improper integral $\int_a^\infty f$.

   (b) Let $p \in \mathbb{R}$, show that $\int_1^\infty x^p \, dx$ exists if and only if $p < -1$.

3. (a) Let $f : [a, \infty) \to \mathbb{R}$ be a function such that $f \in R[a, b]$ for all $b > a$. Show that $\int_a^\infty f$ exists if and only if $\forall\, \epsilon > 0$, there exists $K > a$ such that for all $x, y > K$, $\int_x^y f < \epsilon$.

   (b) Let $f, g : [a, \infty) \to \mathbb{R}$ be two function such that $f, g \in \ R[a, b]$ for all $b > a$ and $0 \leq f \leq g$ on $[a, \infty)$.. Show that $\int_a^\infty f$ exists if $\int_a^\infty g$ exists.

4. (a) Show that $\int_1^\infty \frac{\sin x}{x}$ exists .

   (b) Show that $\int_1^\infty \frac{|\sin x|}{x}$ does not exists .

5. (a) Let $a < b$. Suppose $f : (a, b] \to \mathbb{R}$ satisfies $f \in R[c, b]$ for all $c \in (a, b]$. Define the improper integral $\int_a^b f$.

   (b) Let $f : (0, 1] \to \mathbb{R}$ be a continuous function. Suppose there exists $C > 0$ and $p > -1$ such that $|f(x)| \leq Cx^p$ for all $x \in (0, 1]$. Show that $\int_0^1 f$ exists.

MATH 2060 Mathematical Analysis II
Tutorial Class 8
Lee Man Chun

1. (a) Define pointwise and uniform convergence of a sequence of functions.

   (b) Let $A \subset \mathbb{R}$ and $f_n, f : A \to \mathbb{R}$. Show that f_n does not converge uniformly to f on A if and only if there exists $\epsilon_0 > 0$, a subsequence $\{f_{n_k}\}$ and a sequence $\{x_k\}$ in A such that $|f_{n_k}(x_k) - f(x_k)| \geq \epsilon_0$ for all $k \in \mathbb{N}$.

   (c) Show that the convergence of $f_n(x) = x + \dfrac{x^2}{n}$ is not uniform on $\mathbb{R}$.

   (d) Show that the convergence of $f_n(x) = x + \dfrac{nx}{1 + nx^2}$ is not uniform on $[0, \infty)$.

2. (a) Let $f_n, f : A \to \mathbb{R}$. Show that $\{f_n\}$ converge uniformly to f on A if and only if

$$\sup\{|f_n(x) - f(x)| : x \in A\} \to 0 \text{ as } n \to 0.$$

   (b) If each f_n is continuous on A, show that f is also continuous on A.

   (c) Show that $f_n(x) = \dfrac{x}{1 + nx^2}$ converge uniformly on $\mathbb{R}$.

3. Let $f_n, f : A \to \mathbb{R}$. Suppose each f_n is uniformly continuous on A and $\{f_n\}$ converge uniformly to f on A.

   (a) Show that f is uniformly continuous on A.

   (b) Prove that for all $\epsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in A$, if $|x - y| < \delta$, then $|f_n(x) - f_n(y)| < \epsilon$ for all $n \in \mathbb{N}$

4. Let $f_n, f : [a, b] \to \mathbb{R}$ such that $\{f_n\}$ converge to f pointwisely on $[a, b]$. Suppose each f_n is differentiable, f is continuous and there exist $M > 0$ such that $|f_n'| < M$ on $[a, b]$ for all n, prove that $\{f_n\}$ converge to f uniformly. (Pastpaper 2004-2005)

MATH 2060 Mathematical Analysis II
Tutorial Class 9
Lee Man Chun

1. (a) Prove that if $\{f_n\}$ be a sequence of Riemann integrable function on $[a,b]$ and f_n converge uniformly to f on $[a,b]$, then $f \in R[a,b]$ and $\displaystyle\int_a^b f = \lim_n \int_a^b f_n$.

   (b) Let $\{f_n\}$ be a sequence of functions that converges uniformly to f on A and that satisfies $|f_n(x)| \leq M$ for all $n \in \mathbb{N}$ and all $x \in A$. If g is continuous on $[-M,M]$, show that $\{g \circ f_n\}$ converges uniformly to $g \circ f$ on A.

2. Given an example of sequence of Riemann integrable functions $\{f_n\}$ on $[0,1]$ converging pointwisely to f on $[0,1]$ such that

   (a) $f \in R[0,1]$ but $\displaystyle\lim_{n\to\infty} \int_0^1 f_n \neq \int_0^1 f$.

   (b) f is bounded but f is not Riemann integrable on $[0,1]$.

3. Give an example of sequence of functions (fn) on $[0,1]$ satisfying

   (a) for all n, f_n is discontinuous at any point of $[0,1]$, but f_n converge uniformly to a continuous function f on $[0,1]$.

   (b) $\{f_n\}$ converge pointwisely to f on $[0,1]$ but the convergence is not uniform on any subinterval of $[0,1]$.

4. (a) State the Bounded Convergence Theorem.

   (b) Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Give a sequence of continuous function $\{g_n\}$ on $[a,b]$ such that $|g_n| \leq 1$ on $[a,b]$ and $\{fg_n\}$ converge pointwisely to $|f|$ on $[a,b]$.

   (c) Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Suppose $\int_a^b fg \leq 1$ for all continuous function g on $[a,b]$, prove that $\int_a^b |f| \leq 1$.

5. Let $f_n \in C^1([a,b]), n \in \mathbb{N}$. Show that if f_n' converge uniformly to some function φ on $[a,b]$ and there exists a point $x_0 \in [a,b]$ for which $\{f_n(x_0)\}$ converges, then the sequence of functions $\{f_n\}$ converges uniformly on $[a,b]$ to some function $f \in C^1([a,b])$ and f_n' converges uniformly to $f' = \varphi$.

MATH 2060 Mathematical Analysis II
Tutorial Class 10
Lee Man Chun

1. (a) Suppose $\sum_{n=1}^{\infty} x_n$ converge, show that $x_n \to 0$ and $\sum_{k=n}^{\infty} x_k \to 0$ as n goes to ∞.
   (b) State the Cauchy Criterion for convergence of series.
   (c) Prove the Comparsion Test. i.e. If $\{a_k\}$ and $\{b_k\}$ are two sequences of numbers such that $0 \le a_k \le b_k$ for all $k \in \mathbb{N}$. Then the convergence of $\sum_{n=1}^{\infty} b_n$ implies the convergence of $\sum_{n=1}^{\infty} a_n$.
   (d) Show that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverge and $\sum_{n=1}^{\infty} ne^{-n^2}$ converge.
   (e) Show that for any $\epsilon > 0$, the series $\sum_{n=1}^{\infty} \frac{n}{n^{2+\epsilon} - n + 1}$ converge.

2. (a) Suppose $x_n \ge 0$. Show that $\sum_{n=1}^{\infty} x_n$ converge if and only if its partial sum is bounded.
   (b) Suppose $x_n \ge 0$ and $\sum_{n=1}^{\infty} x_n$ converge. Show that the following series converge:
   (i) $\sum_{n=1}^{\infty} x_n^{1+\epsilon}$ (ii) $\sum_{n=1}^{\infty} \frac{\sqrt{x_n}}{n}$ (iii) $\sum_{n=1}^{\infty} \sqrt{x_n x_{n+1}}$.
   (c) Suppose $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ are series of positive numbers such that

$$\lim_{k\to\infty} \frac{a_k}{b_k} = l, \quad l > 0.$$

   Prove that the series $\sum_{k=1}^{\infty} a_k$ converges if and only if $\sum_{k=1}^{\infty} b_k$ converges.

3. (a) State the Ratio Test for the convergence of series.
   (b) Test the convergence of the series $\sum_{n=1}^{\infty} x_n$ with general term:
   (i) $x_n = (\frac{n}{2n+1})^n$ (ii) $x_n = \frac{3^n}{n^2}$ (iii) $x_n = \frac{n^n}{n!}$.

4. (a) State the Integral Test for convergence of series.
   (b) For $\alpha > 0$, consider the series

$$\sum_{k=1}^{\infty} \frac{1}{(k+1)[\ln(k+1)]^{\alpha}},$$

   Find the values of α at which the series converge.
   (c) Give an exmaple of $x_n > 0$ such that $\lim_{n\to\infty} x_n = 0$ but $\sum_{n=2}^{\infty} \frac{x_n}{n \log n}$ diverge.

MATH 2060 Mathematical Analysis II
Tutorial Class 11
Lee Man Chun

1. Show that the convergence of $\sum_n \sqrt{a_n a_{n+1}}$ does not imply the convergence of $\sum_n a_n$, even if $a_n > 0, \ \forall n \in \mathbb{N}$.

2. If $\{a_n\}$ is a decreasing sequence of strictly positive numbers and if $\sum_n a_n$ is convergent, show that $\lim_{n\to} na_n = 0$.

3. If $a_n \neq 0$ for all $n \in \mathbb{N}$ and
$$\limsup_n |\frac{a_{n+1}}{a_n}| = L.$$
   (a) Prove that if $L < 1$, then the series $\sum a_n$ converges absolutely.
   (b) If $\liminf_n |\dfrac{a_{n+1}}{a_n}| > 1$, show that the series diverges.

4. If
$$\limsup_n |a_n|^{1/n} = L.$$
   (a) Prove that if $L < 1$, then the series $\sum a_n$ converges absolutely.
   (b) Prove that if $L > 1$, then the series $\sum a_n$ diverge.
   (c) If $a_n > 0$, show that
$$\limsup_n a_n^{1/n} \ \leq \ \limsup_n |\frac{a_{n+1}}{a_n}|.$$

5. Determine the convergence of following series.
   (a) $\sum_{n=1}^{\infty} \sin \frac{1}{n}$ (b) $\sum_{n=1}^{\infty} (1 - \cos \frac{1}{n})$ (c) $\sum_{n=1}^{\infty} \frac{(-1)^n \log n}{2n+3}$
   (d) $\sum_{n=1}^{\infty} \frac{1+\log^2 n}{n \log^2 n}$ (e) $\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$ (f) $\sum_{n=1}^{\infty} \frac{\log n}{n+\log n}$

6. Let A be the set of positive integers which do not contain the digit 9 in the decimal expansion. Prove that
$$\sum_{a\in A} \frac{1}{a} \text{ exists.}$$

7. Find the value of $a \in \mathbb{R}$ such that the series
$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \sin \frac{1}{n} \right)^a$$
   exists.

MATH 2060 Mathematical Analysis II
Tutorial Class 12

1. (a) Show that $f(x) = \sum_{n=1}^{\infty} \frac{\cos 3^n x}{2^n}$ is a continuous function on $\mathbb{R}$.
   (b) Prove that $f(x) = \sum_{n=1}^{\infty} \frac{e^{nx}}{n!}$ is a continuous function on $\mathbb{R}$ but the convergence is non-uniform.
   (c) Show that $f(x) = \sum_{n=1}^{\infty} \frac{n^{10}}{x^n}$ is a differentiable function on $(1, \infty)$.

2. Let $\{a_n\}$ be a sequence such that $\sum_{n=1}^{\infty} n|a_n|$ converge. Show that $f(x) = \sum_{n=1}^{\infty} a_n \sin nx$ converge on $\mathbb{R}$ and $f'(x) = \sum_{n=1}^{\infty} na_n \cos nx$.

3. Show that the convergence of $\sum_{n=1}^{\infty} \frac{\sin nx}{n}$ is not uniform on $[0, 1]$.

4. (a) State the Cauchy-Hadmand Theorem for power series.
   (b) Suppose a power series $\sum a_n x^n$ converge at some $x_0 \in \mathbb{R}$. Show that it converge absolutely for all $|x| < |x_0|$.
   (c) Suppose a power series converge absolutely at some $c \in \mathbb{R}$, show that it converge uniformly on the interval $[-c, c]$.

5. Find the radius of convergence R of the following series:
   (i) $\sum \frac{2^n}{n^2} x^n$ (ii) $\sum n! x^n$ (iii) $\sum \frac{n!}{(2n)!} x^n$ (iv) $\sum \frac{(-1)^n + 2^n}{3^n} x^n$.

6. (a) Prove that for all $x \in (-1, 1)$,
      i. $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n$,
      ii. $\log(1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$ and
      iii. $\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$.
   (b) Find the value of $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$.

7. Let $f_n : [a, b] \to \mathbb{R}$ such that $\sum f_n$ converge uniformly on (a, b). Suppose $\lim_{x \to a^+} f_n(x) = c_n \in \mathbb{R}$. Show that $\sum c_n$ converge and

$$\lim_{x \to a^+} \sum f_n(x) = \sum c_n.$$

past paper question:

Suppose the series $\sum a_n x^n$ has radius of convergence one. Let $f(x) = \sum a_n x^n$, $x \in (-1, 1)$. If $[a, b] \subset (0, 1)$ and $f_n(x) \doteq f(x - \frac{1}{n}), x \in [a, b]$, show that $f_n \to f$ uniformly on $[a, b]$.